

ON THE SCHWARZ-CHRISTOFFEL TRANSFORMATION AND p -VALENT FUNCTIONS

BY

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1. **Introduction.** It is well known⁽¹⁾ that the function

$$(1.1) \quad w = f_1(z) = c_1 \int_0^z \prod_{j=1}^m (1 - z_j t)^{-\gamma_j} dt + c_2$$

subject to the conditions

$$(1.2) \quad |z_j| = 1, \quad j = 1, 2, \dots, m,$$

$$(1.3) \quad \sum_{j=1}^m \gamma_j = 2,$$

$$(1.4) \quad 0 < \gamma_j \leq 2, \quad j = 1, 2, \dots, m,$$

maps the open unit circle $|z| < 1$ (hereafter denoted by E) onto P the interior of an m -sided convex polygon. The vertices of the polygon are $w_j = f_1(\bar{z}_j)$ and the exterior angle⁽²⁾ at the vertex w_j is $\gamma_j \pi$. Conversely if P is given, then $z_1, z_2, \dots, z_m, c_1$, and c_2 can be determined such that (1.1) maps E onto P , and moreover the origin can be carried into any preassigned point of P and the value of $\arg f'(0)$ can be arbitrarily preassigned. The equation (1.1) subject to the conditions (1.2) and (1.3) is one form of the Schwarz-Christoffel transformation⁽³⁾.

Schwarz⁽⁴⁾ stated that the formula (1.1) is easily generalized to the case where P is a multi-sheeted domain bounded by straight lines and containing branch points, and Christoffel⁽⁵⁾ considered this generalization in some detail.

Study⁽⁶⁾, Loewner⁽⁷⁾, Gronwall⁽⁸⁾, Bieberbach⁽⁹⁾, Paatero⁽¹⁰⁾, and

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⁽¹⁾ Churchill, *Introduction to complex variables and applications*, New York, McGraw-Hill, 1948.

⁽²⁾ If $1 \leq \gamma_j \leq 2$, then $w_j = \infty$. There is no difficulty in extending the concept of an exterior angle to this case. The region P is unbounded but still convex.

⁽³⁾ The Schwarz-Christoffel transformation is usually given as a function which maps the upper half-plane onto the interior of a polygon. It is easy to obtain (1.1) from the standard form as indicated in §2.

⁽⁴⁾ *Ueber einige Abbildungsaufgaben*, J. Reine Angew. Math. vol. 70 (1869) pp. 105-120, or *Mathematische Abhandlungen*, vol. 2, pp. 65-83, in particular p. 77.

⁽⁵⁾ *Ueber die Abbildung einer n -blattrigen einfach Zusammenhängender ebenen Fläche auf einen Kreise*, Göttingen Nachrichten, 1870, pp. 359-369.

⁽⁶⁾ *Vorlesungen über ausgewählten Gegenstände der Geometrie*, vol. 2, Leipzig, Teubner, 1913.

⁽⁷⁾ *Untersuchungen über die Verzerrung bei konformen Abbildungen des Einheitskreises*

Robertson⁽¹¹⁾ have used the Schwarz-Christoffel transformation as a starting point for the derivation of properties of univalent functions. As far as I have been able to discover, it was Robertson who first pointed out that equation (1.1) leads to a very simple proof that $|b_n| \leq n|b_1|$ for the coefficients of a univalent function in the special case that the image of E is starlike with respect to the origin.

By using Robertson's methods together with a generalization of (1.1) we are able to prove a number of theorems about certain subclasses of the class of p -valent functions.

As a by-product, we obtain two more proofs that

$$(1.5) \quad \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6},$$

and we prove the arithmetic identities

$$(1.6) \quad D' = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} = 2 \sum_{m=2}^{\infty} \frac{1}{(2m-1)^2} H_{2m-2},$$

$$(1.7) \quad E' = \sum_{m=1}^{\infty} \frac{1}{(2m)^3} = \frac{2}{9} \sum_{m=1}^{\infty} \frac{1}{(2m)^2} H_{2m-1},$$

$$(1.8) \quad F' = \sum_{m=1}^{\infty} \frac{1}{m^3} = \sum_{m=2}^{\infty} \frac{1}{m^2} H_{m-1},$$

where

$$(1.9) \quad H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}.$$

2. The generalized Schwarz-Christoffel transformation. The material of this paragraph is either contained or implied in the works of Schwarz and Christoffel. It is included here for completeness. Let

$$(2.1) \quad g(u) = \prod_{j=1}^m (u_j - u)^{-\gamma_j}, \quad u_1 < u_2 < \cdots < u_m,$$

$|z| < 1$, die durch Funktionen mit nicht verschwindender Ableitung geliefert werden, *Berichte der Gesellschaft die Wissenschaften zu Leipzig* vol. 69 (1917) pp. 89–106.

⁽⁸⁾ *Sur la déformation dans la représentation conforme*, *C. R. Acad. Sci. Paris* vol. 162 (1916) pp. 249–252.

⁽⁹⁾ *Aufstellung und Beweis des Drehungssatzes für schlichte konforme Abbildungen*, *Math. Zeit.* vol. 4 (1919) pp. 295–305.

⁽¹⁰⁾ *Über die conforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind*, *Annales Academiae Scientiarum Fennicae*, ser. A vol. 33 (1931) pp. 1–78.

⁽¹¹⁾ *On the theory of univalent functions*, *Ann. of Math.* vol. 37 (1936) pp. 374–408, in particular p. 380.

$$(2.2) \quad h(u) = \prod_{j=1}^{p-1} (\alpha_j - u)(\bar{\alpha}_j - u), \quad \Im(\alpha_j) > 0,$$

$$(2.3) \quad w = f_2(z) = c_3 \int_i^z g(u) h(u) du + c_4,$$

where the path of integration is subject to the restrictions $\Im(u) \geq 0$, $u \neq u_j$, $j=1, 2, \dots, m$. When $\Im(u)=0$, $\arg g(u)$ is constant and $h(u)>0$. So $f_2(z)$ maps each segment $u < u_1$, $u_j < u < u_{j+1}$, $u_n < u$, onto some straight line segment. These image segments may be half-rays extending to infinity, or may be full lines. The function $f_2(z)$ is regular in $\Im(z)>0$, and has critical points at $z=\alpha_j$. Thus $f_2(z)$ maps $\Im(z)>0$ onto a multi-sheeted region whose boundary consists only of straight line segments, half-rays, and full lines. We shall refer to such regions as multi-sheeted polygons. The function $f_2(z)$ will be regular and univalent in a neighborhood of infinity if we require that

$$(2.4) \quad \sum_{j=1}^m \gamma_j = 2p.$$

Conversely⁽¹²⁾, if P is any multi-sheeted polygon subject only to the condition that there exists a function $f(z)$ mapping $\Im(z)>0$ onto P , regular for $\Im(z)>0$, and regular and univalent in a neighborhood of infinity, then $f(z)$ has the form (2.3) and (2.4) is satisfied⁽¹³⁾.

The substitution $u=i(1-t)/(1+t)$ in the integral (2.3) gives

$$(2.5) \quad w = f(z) = c_5 \int_0^z \prod_{j=1}^m (1 - z_j t)^{-\gamma_j} \prod_{j=1}^{p-1} (t - \beta_j)(1 - \bar{\beta}_j t) dt + c_4,$$

where

$$(2.6) \quad |z_j| = 1, \quad z_j \neq z_k \quad \text{if} \quad j \neq k; j, k = 1, 2, \dots, m$$

$$(2.7) \quad |\beta_j| < 1, \quad j = 1, 2, \dots, p-1.$$

Since $u=i(1-t)/(1+t)$ maps E onto the half-plane $\Im(u)>0$, $w=f(z)$ maps E onto a multi-sheeted polygon P . The vertices of P are $w_j=f(\bar{z}_j)$ and the exterior angle at w_j is $\gamma_j\pi$. P has branch points at $w_j^*=f(\beta_j)$, and the number of sheets tied at w_j^* is just one more than the number of times $(t-\beta_j)$ occurs as a factor in the integrand of (2.5). Again if P is any multi-sheeted polygon subject only to the condition that there exists a function $f(z)$, regular in E and mapping E onto P , then $f(z)$ has the form (2.5) and (2.4) is satisfied.

⁽¹²⁾ We omit the proof of this. It is an elementary generalization of the proof in the plane case. See Julia, *Leçons sur la représentation conforme des aires simplement connexes*, Gauthier-Villars, 1931, pp. 67-71.

⁽¹³⁾ A still more general form of the Schwarz-Christoffel transformation was obtained independently by D. Gilbarg, *A generalization of the Schwarz-Christoffel transformation*, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 609-612.

3. Some examples. The function

$$(3.1) \quad \begin{aligned} w = f(z) &= \int_0^z \frac{(t-a)(1-at)}{1-t^4} dt \quad (-1 < a < 1) \\ &= \frac{1}{4} \left\{ (1+a^2) \log \frac{1+z^2}{1-z^2} - 2a \log \frac{1+z}{1-z} \right\} \end{aligned}$$

maps E onto the two-sheeted region comprised of the two infinite strips $|\Im(w)| < \pi(1-a)^2/8$ and $|\Im(w)| < \pi(1+a)^2/8$. It is perhaps interesting to observe the limit case $a=1$. One of the strips disappears and the limit function maps E onto the infinite strip $|\Im(w)| < \pi/2$, slit along the real axis from $-\infty$ to $-2^{-1} \log 2$, the latter being the limit point of the branch point $f(a)$ of (3.1) as $a \rightarrow 1$.

Similarly the function

$$(3.2) \quad \begin{aligned} w &= \int_0^z \frac{(t-a)(1-at)}{1+t^4} dt \quad (-1 < a < 1) \\ &= \frac{i}{4} \left\{ a2^{1/2} \log \frac{1+i2^{1/2}z-z^2}{1-i2^{1/2}z-z^2} + (1+a^2) \log \frac{1-iz^2}{1+iz^2} \right\} \end{aligned}$$

maps E onto a region which consists of the two infinite strips $-\pi(1+a^2)/8 < \Re(w) < \pi(1+2^{3/2}a+a^2)/8$ and $-\pi(1+a^2)/8 < \Re(w) < \pi(1-2^{3/2}a+a^2)/8$. The symmetrical position of the two strips in the first example is lacking in the second example.

As a third example, the function

$$(3.3) \quad \begin{aligned} w &= \int_0^z \frac{(t-a)(1-at)(t-b)(1-bt)}{1-t^6} dt \quad (-1 < a, b < 1) \\ &= \frac{1}{6} \left\{ (1+a^2+b^2+ab+a^2b^2) \log \frac{1+z^3}{1-z^3} + 3ab \log \frac{1+z}{1-z} \right. \\ &\quad \left. + (a+b)(1+ab) \log \frac{1-2z^2+z^4}{1+z^2+z^4} \right\} \end{aligned}$$

maps E onto a region comprised of the three infinite strips

$$|\Im(w)| < \frac{\pi}{12} (1+a)^2(1+b)^2,$$

$$|\Im(w)| < \frac{\pi}{12} \{ (1+a^2)(1+b^2) - 2ab \},$$

$$|\Im(w)| < \frac{\pi}{12} (1-a)^2(1-b)^2.$$

If we use the infinite series form of the function, multiply by p , and introduce the new variable $\zeta = 2p/n$,

$$(4.2) \quad 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \zeta}{m!(2m + \zeta)} \prod_{k=0}^{m-1} (\zeta + k) \\ = \cos(\zeta\pi/2) \left\{ 1 + \sum_{m=1}^{\infty} \frac{\zeta}{m!(2m + \zeta)} \prod_{k=0}^{m-1} (\zeta + k) \right\}.$$

Each side is an analytic function of ζ , for a sufficiently restricted ζ , and since the two functions coincide on the everywhere dense set of rationals $0 < \zeta = 2p/n < 1$, (4.2) is an identity in ζ and we may equate coefficients of like powers of ζ in the power series expansion. For ζ^2 this gives

$$(4.3) \quad \frac{\pi^2}{8} = \sum_{m=0}^{\infty} \frac{1}{(2m + 1)^2},$$

from which, by a well known trick, one can obtain (1.5). Equating coefficients of ζ^3 gives (1.6). Other identities can be obtained by using the coefficients of higher powers of ζ , but these appear to be quite complicated and of little interest.

By dissecting the regular p -sheeted n -gon into $2n$ triangles, it is easy to see that the area of that figure is

$$(4.4) \quad A = nf(1) \left| f(e^{i\pi/n}) \right| \sin(p\pi/n).$$

On the other hand, by a well known formula⁽¹⁴⁾ for the area of the image of E , applied to (3.5), we have

$$(4.5) \quad A = \pi \sum_{m=1}^{\infty} m |a_m|^2 = \pi \left\{ \frac{1}{p} + \sum_{m=1}^{\infty} \frac{1}{(mn + p)(m!)^2} \prod_{k=0}^{m-1} \left(\frac{2p}{n} + k \right)^2 \right\}.$$

If we equate (4.4) and (4.5), multiply by $2p^2/n$, and introduce the new variable ζ , we find

$$(4.6) \quad \pi \left\{ \zeta + \sum_{m=1}^{\infty} \frac{\zeta^2}{(2m + \zeta)(m!)^2} \prod_{k=0}^{m-1} (\zeta + k)^2 \right\} \\ = 2 \sin(\zeta\pi/2) \left\{ 1 + \sum_{m=1}^{\infty} \frac{\zeta}{(2m + \zeta)m!} \prod_{k=0}^{m-1} (\zeta + k) \right\} \\ \cdot \left\{ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \zeta}{(2m + \zeta)m!} \prod_{k=0}^{m-1} (\zeta + k) \right\}.$$

Again we have coincidence of two analytic functions of ζ for an everywhere dense set of rational ζ , in the interval $0 < \zeta < 1$, and hence (4.6) is an identity. If we equate coefficients of ζ^3 in the power series expansions of (4.6) we have

⁽¹⁴⁾ Pólya-Szegő, *Aufgaben und Lehrsätze*, vol. 1, New York, Dover, 1945, p. 109.

$$(4.7) \quad \frac{\pi^3}{12} = 2\pi \sum_{m=1}^{\infty} \frac{1}{(2m)^2},$$

and hence (1.5). If we equate coefficients of ζ^4 , we obtain (1.7). Finally we can combine (4.1) and (4.4) to obtain

$$(4.8) \quad A = n\{f(1)\}^2 \cos(p\pi/n) \sin(p\pi/n) = 2^{-1}n\{f(1)\}^2 \sin(2p\pi/n).$$

This together with (4.5) yields

$$(4.9) \quad \pi \left\{ \zeta + \sum_{m=1}^{\infty} \frac{\zeta^2}{(2m+\zeta)(m!)^2} \prod_{k=0}^{m-1} (\zeta+k)^2 \right\} \\ = \sin \zeta \pi \left\{ 1 + \sum_{m=1}^{\infty} \frac{\zeta}{(2m+\zeta)m!} \prod_{k=0}^{m-1} (\zeta+k) \right\}^2,$$

once more an identity in ζ . Equating coefficients of ζ^4 , we obtain (1.8).

The three equations (1.6), (1.7), and (1.8) are certainly not independent. For if we replace the multipliers 2, $2/9$, and 1 by unknowns, d , e , and f , it is easy to see that

$$(4.10) \quad \frac{D'}{d} + \frac{E'}{e} = \frac{F'}{f},$$

and since $E' = F'/8$ and $D' = 7F'/8$,

$$(4.11) \quad \frac{7}{d} + \frac{1}{e} = \frac{8}{f}.$$

Thus any two of the three equations (1.6), (1.7), and (1.8) would imply the third.

5. The two subclasses of p -valent functions. We generalize the idea of a convex region to include certain p -sheeted regions in the following way. Let w_b be a boundary point of a p -sheeted region R and let $w_b(r)$ be the subregion of R consisting of those points w of R in the same sheet with w_b and satisfying the inequality $|w - w_b| < r$. If for every boundary point w_b of R there is an $r > 0$ such that $w_b(r)$ is convex, then R is said to be a locally convex region.

Now if $f(z)$ is regular in E and if $f'(re^{i\theta}) \neq 0$, it will map $|z| < r < 1$ onto a region $R(r)$ whose boundary $f(re^{i\theta})$ is an analytic curve with a continuously turning tangent. Let ψ be the angle of intersection of this tangent with the real axis. The angle ψ is not uniquely determined as a function of θ , but will be so if we fix on one of the possible values of ψ when $\theta = 0$, and determine $\psi(\theta)$ for $0 \leq \theta < 2\pi$ by continuity. Then $\psi'(\theta) \geq 0$ if and only if $R(r)$ is a locally convex region. It is well known⁽¹⁵⁾ that $\psi'(\theta) = 1 + \Re(zf''(z)/f'(z))$.

⁽¹⁵⁾ Montel, *Leçons sur les fonctions univalentes ou multivalentes*, Gauthier-Villars, 1933, pp. 11-14.

DEFINITION. The function $f(z)$ is said to be an element of the class $C(p)$, p a positive integer, if it is regular in E , if $f(0)=0$, and if there is a $\rho < 1$ such that for all r in the interval $\rho < r < 1$

$$(5.1) \quad G(r, \theta) = 1 + \Re \left(r e^{i\theta} \frac{f''(r e^{i\theta})}{f'(r e^{i\theta})} \right) > 0, \quad 0 \leq \theta \leq 2\pi,$$

and

$$(5.2) \quad \int_0^{2\pi} G(r, \theta) d\theta = 2\pi p.$$

If $f(z) \in C(p)$, it maps $|z| < r < 1$ onto a locally convex region. Furthermore $f'(z)$ has exactly $p-1$ roots in the circle $|z| < r$, multiple roots being counted in accordance with their multiplicities. For if ν is the number of these roots,

$$\begin{aligned} 2\pi p &= \int_0^{2\pi} G(r, \theta) d\theta = \Re \oint_{z=r e^{i\theta}} \left(1 + \frac{z f''(z)}{f'(z)} \right) dz \\ &= 2\pi + \Re \frac{1}{i} \oint \frac{f''(z)}{f'(z)} dz = 2\pi + 2\pi \nu. \end{aligned}$$

Finally $f(z)$ is at most p -valent in E . For the contour integral which gives the number of roots of $f(z) - c$ is just the variation of $\arg(f(z) - c)$ along the contour, divided by 2π . But the bounding curve of $R(r)$ for a function of class $C(p)$ is a curve which turns continuously in a counterclockwise manner as θ runs from 0 to 2π , and the total number of complete turns is exactly p . Thus the variation in $\arg(f(z) - c)$ cannot exceed $2\pi p$. It is obvious that $f(z) \in C(p)$ may be divalent.

To generalize the concept of a plane region starlike with respect to a point, to include certain p -sheeted regions, we consider the line joining a boundary point w_b with the given point. If as w_b describes the boundary of R , the line turns continuously in a counterclockwise direction (or continuously in a clockwise direction) then R is said to be starlike with respect to the given point⁽¹⁶⁾.

Now let $F(z)$ be regular in E and $F(r e^{i\theta}) \neq 0$ for $\rho < r < 1$. If $\phi = \arg F(r e^{i\theta})$, ϕ is not uniquely determined as a function of θ , but will be so if we fix on one of the possible values of ϕ when $\theta = 0$, and determine $\phi(\theta)$ for $0 \leq \theta < 2\pi$ by continuity. Then $\phi'(\theta) > 0$ if and only if $R(r)$ is starlike with respect to the origin. It is well known⁽¹⁵⁾ that $\phi'(\theta) = \Re(z F'(z)/F(z))$.

⁽¹⁶⁾ This concept of a generalized starlike function has been used previously by the following authors: Obrechhoff, Bull. Sci. Math. (2) vol. 60 (1935) pp. 36-42; Ozaki, Science Reports of the Tokyo Bunrika Daigaku Section A, 2 No. 32 and 36 (1936) and 4 No. 77 (1941); Robertson, Ann. of Math. vol. 38 (1937) pp. 770-783 and vol. 42 (1941) pp. 829-838, Duke Math. J. vol. 12 (1945) pp. 669-684; Biernacki, Mathematica Timisoara vol. 23 (1949) pp. 54-59.

DEFINITION. The function $F(z)$ is said to be an element of the class $S(p)$, p a positive integer, if it is regular in E , if $F(0)=0$, and if there is a ρ such that for all r in the interval $\rho < r < 1$

$$(5.3) \quad H(r, \theta) = \Re \left(\frac{re^{i\theta} F'(re^{i\theta})}{F(re^{i\theta})} \right) > 0, \quad 0 \leq \theta \leq 2\pi,$$

and

$$(5.4) \quad \int_0^{2\pi} H(r, \theta) d\theta = 2\pi p.$$

If $F(z) \in S(p)$, it maps $|z| < r < 1$ onto a region starlike with respect to the origin. Just as (5.2) implied that $f'(z)$ had $p-1$ roots in E , the condition (5.4) implies that $F(z)$ has p roots in E . To see that $F(z)$ is not more than p -valent in E consider $\delta(\lambda) = \Delta \arg (F(re^{i\theta}) - \lambda c)$, where Δ denotes the change as θ runs from 0 to 2π . Let c be fixed and let λ vary from 0 to 1. We already have $\delta(0) = 2\pi p$. But $\delta(\lambda)$ is always an integer multiple of 2π , and is a continuous function of λ except when $\lambda c = w_b$, a boundary point of the image of $|z| < r$. At such points $\delta(\lambda)$ jumps $\pm 2\pi$. By (5.3) and (5.4) there are exactly p such boundary points for every c . Finally we remark that for $|c|$ sufficiently large, $\delta(1) = 0$, so that the jumps in $\delta(\lambda)$ must be -2π and $0 \leq \delta(\lambda) \leq 2\pi p$ for all λ and c .

For $p=1$, $C(p)$ and $S(p)$ are the classical univalent functions, convex and starlike respectively, and $S(1) \supset C(1)$. It is worth noting that for $p \geq 2$, $S(p) \not\supset C(p)$.

LEMMA 1⁽¹⁷⁾. Let $c \neq 0$ be an arbitrary constant. If $f(z) \in C(p)$, then $F(z) = czf'(z) \in S(p)$, and conversely if $F(z) \in S(p)$, then $f(z) \in C(p)$, where

$$(5.5) \quad f(z) = c \int_0^z \frac{F(t)}{t} dt.$$

Proof. For $f(z)$ and $F(z)$ related as indicated,

$$(5.6) \quad z \frac{F'(z)}{F(z)} = z \frac{cf'(z) + czf''(z)}{czf'(z)} = 1 + z \frac{f''(z)}{f'(z)},$$

so that (5.3) implies (5.1) and (5.4) implies (5.2), and conversely.

LEMMA 2. Let $f(z) = z^q + \dots$, with critical points $\beta_1, \beta_2, \dots, \beta_{p-q} \neq 0$ in E , be an element of $C(p)$, and suppose further that $f(z)$ is regular for $|z| = 1$. There is a sequence of functions of the form

⁽¹⁷⁾ First proved in the univalent case by J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. vol. 17 (1915) pp. 12-22. See also Montel, loc. cit.

$$(5.7) \quad f_m(z) = A_m \int_0^z t^{q-1} \prod_{j=1}^m (1 - z_j t)^{-\gamma_j} \prod_{j=1}^{p-q} \left(1 - \frac{t}{\beta_j^{(m)}}\right) (1 - \bar{\beta}_j^{(m)} t) dt$$

with $|z_j| = 1$, $|\beta_j^{(m)}| < 1$, $0 < \gamma_j < 1$ for $j = 1, 2, \dots, m$, and

$$(5.8) \quad \sum_{j=1}^m \gamma_j = 2p,$$

such that, as $m \rightarrow \infty$, $\beta_j^{(m)} \rightarrow \beta_j$, $A_m \rightarrow q$, and $f_m(z) \rightarrow f(z)$ uniformly for $|z| \leq r < 1$.

The proof of this lemma is analogous to the one given by Robertson⁽¹⁸⁾ and will be omitted. We can apply this lemma to functions of class $C(p)$ since if $f(z) \in C(p)$, then $f(rz)/r^q$ will satisfy the conditions for the lemma for every r , $\rho < r < 1$. Finally in view of the convergence properties we may consider not (5.7) but the simplified version

$$(5.9) \quad f_m(z) = q \int_0^z t^{q-1} \prod_{j=1}^m (1 - z_j t)^{-\gamma_j} \prod_{j=1}^{p-q} \left(1 - \frac{t}{\beta_j}\right) (1 - \bar{\beta}_j t) dt.$$

LEMMA 3. *Let*

$$\prod_{j=1}^m (1 - z_j t)^{-\gamma_j} = 1 + \sum_{n=1}^{\infty} c_n t^n,$$

where z_j and γ_j are subject to the conditions of Lemma 2. Then $|c_n| \leq C_{n+2p-1, 2p-1}$, with equality if and only if $z_1 = z_2 = \dots = z_m$.

Proof. Clearly $c_n = c_n(z_1, z_2, \dots, z_m)$ is a homogeneous polynomial of n th degree with positive coefficients. Hence a maximum occurs when all z_j are equal. The value of the maximum is easily obtained by setting all $z_j = 1$. Thus $C_{n+2p-1, 2p-1}$ is just the sum of the coefficients of the polynomial. To see that this is the only case in which equality occurs, suppose without loss of generality that $z_1 = 1$ and $z_k = e^{i\theta} \neq 1$. Since $c_n(z_1, z_2, \dots, z_m)$ contains the terms z_1^n and $z_1^{n-1} z_k$ both with positive coefficients, it follows that in this case $|c_n| < C_{n+2p-1, 2p-1}$.

THEOREM 1. *Let $f(z) \in C(p)$, of the form*

$$(5.10) \quad f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n, \quad 1 \leq q \leq p,$$

having $p-q$ critical points $\beta_1, \beta_2, \dots, \beta_{p-q} \neq 0$ in E . Then

$$(5.11) \quad |f(re^{i\theta})| \leq f_M(r), \quad 0 \leq r < 1,$$

and

⁽¹⁸⁾ Loc. cit. footnote 11, pp. 376-377.

$$(5.12) \quad |a_n| \leq A_n, \quad n = q+1, q+2, \dots,$$

where

$$(5.13) \quad \begin{aligned} f_M(z) &= q \int_0^z \frac{t^{q-1}}{(1-t)^{2p}} \prod_{j=1}^{p-q} \left(1 + \frac{t}{|\beta_j|}\right) (1+t|\beta_j|) dt \\ &= z^q + \sum_{n=q+1}^{\infty} A_n z^n. \end{aligned}$$

The bounds (5.11) and (5.12) are sharp, since $f_M(z) \in C(p)$. The extremal function $f_M(z)$ maps E onto a region R_M consisting of $p-1$ full planes and a half-plane, $\Re(w) < f_M(-1)$ if p is odd, $\Re(w) > f_M(-1)$ if p is even.

Proof. The function $f(z)$ may be approximated by a sequence of functions of the form (5.9). Without loss of generality set $z_1=1$. Then by Lemma 3, the maximum coefficients in the power series for the first product in (5.9) occur when $z_2=z_3=\dots=z_m=1$. These coefficients are then positive, and hence in combining the second product with the first, maximal coefficients are obtained by replacing β_j by $-|\beta_j|$ for $j=1, 2, \dots, p-q$. Then (5.9) becomes (5.13) and the inequality (5.12) is established. The inequality (5.11) can be obtained by a similar argument, but it is simpler to observe that (5.11) is a consequence of (5.12), since all the coefficients of $f_M(z)$ are positive.

Since $G(1, \theta) = 0$ for $f_M(z)$ and since further $f'_M(z)$ does not vanish on $|z|=1$, the boundary of R consists of a single straight line. The reality of the coefficients implies that the line is symmetric about the real axis, that is, orthogonal to the real axis. Finally by noting that $f'_M(-1)$ has the sign of $(-1)^{p-1}$, the position of the half-plane is easily determined.

THEOREM 2. Let $F(z) \in S(p)$ of the form (5.10) have $p-q$ roots $\beta_1, \beta_2, \dots, \beta_{p-q} \neq 0$ in E . Then

$$(5.14) \quad |F(re^{i\theta})| \leq F_M(r), \quad 0 \leq r < 1,$$

and

$$(5.15) \quad |a_n| \leq B_n, \quad n = q+1, q+2, \dots,$$

where

$$(5.16) \quad F_M(z) = \frac{z^q}{(1-z)^{2p}} \prod_{j=1}^{p-q} \left(1 + \frac{z}{|\beta_j|}\right) (1+z|\beta_j|) = z^q + \sum_{n=q+1}^{\infty} B_n z^n.$$

The bounds (5.14) and (5.15) are sharp, since $F_M(z) \in S(p)$. The extremal function $F_M(z)$ maps E onto a region R_M consisting of $p-1$ full planes and one plane with a single radial slit.

Proof. The inequality (5.15) follows from (5.12) by applying Lemma 1 and Theorem 1 to the function $F(z) = zf'(z)/q$, where $f(z)$ satisfies the conditions of Theorem 1. The inequality (5.14) is a consequence of (5.15). Since $H(1, \theta) = 0$ for $F_M(z)$ the boundary of R_M consists of radial lines. But the only root of $F'_M(z)$ on $|z| = 1$ is the simple root at $z = -1$, and the only singularity is the pole at $z = +1$. Hence the boundary of R_M consists of a single radial line.

There are some special cases of these two theorems which are worth mentioning. Let us suppose that instead of fixing $|\beta_j|$ for the critical points of $f(z)$ or the roots of $F(z)$, we merely require that there is a $\rho > 0$ such that $|\beta_j| \geq \rho$ for $j = 1, 2, \dots, p - q$. Then since $|\bar{\beta}_j + \beta_j^{-1}| \leq |\beta_j| + |\beta_j|^{-1} \leq \rho + \rho^{-1}$, it is easy to see that the extremalizing functions of Theorems 1 and 2 must be replaced by

$$(5.17) \quad f_M(z) = q \int_0^z \frac{t^{q-1}}{(1-t)^{2p}} \left(1 + \frac{t}{\rho}\right)^{p-q} (1 + \rho t)^{p-q} dt,$$

and

$$(5.18) \quad F_M(z) = \frac{z^q}{(1-z)^{2p}} \left(1 + \frac{z}{\rho}\right)^{p-q} (1 + \rho z)^{p-q},$$

with the same conclusions holding.

In the special case that $q = p$, we have

$$(5.19) \quad f_M(z) = p \int_0^z \frac{t^{p-1}}{(1-t)^{2p}} dt = \sum_{n=p}^{\infty} \frac{p}{n} C_{n+p-1, 2p-1} z^n$$

and

$$(5.20) \quad F_M(z) = \frac{z^p}{(1-z)^{2p}} = \sum_{n=p}^{\infty} C_{n+p-1, 2p-1} z^n,$$

again with the same conclusions holding.

This last special case does not require the Schwarz-Christoffel transformation for the proof. For any function $F(z) = z^p + \dots \in S(p)$ can be expressed as $\{G(z)\}^p$, where $G(z) = z + \dots \in S(1)$, and since $z/(1-z)^2$ is the extremalizing function for $S(1)$, (5.20) follows. This bound $|a_n| \leq C_{n+p-1, 2p-1}$ was obtained previously by Robertson⁽¹⁹⁾ for a larger class of functions.

Each of the integrals (5.13), (5.17), and (5.19) may be expressed in terms of a finite number of the elementary functions. However, the last one (5.19) can be obtained in a simple way directly from the mapping properties of the function.

(19) *A representation of all analytic functions in terms of functions with positive real parts*, Ann. of Math. vol. 38 (1937) pp. 770-783, in particular p. 778, inequality (5.8). See also *Star center points of multivalent functions*, Duke Math. J. vol. 12 (1945) pp. 669-684, in particular p. 681, inequality (6.9).

For simplicity let p be odd and let $s(z)$ be defined by the following properties. $s(z)$ maps E onto a region S consisting of $p-1$ full planes and a half-plane $\Re(w) > -1$, all tied at $w=0$. Thus $s(z)$ has a p th order root at $z=0$. Let $s^{(p)}(0) > 0$. The function $s(z)$ is now determined uniquely. The symmetry of S about the real axis shows that the interval $-1 < z < 1$ goes into a real segment, and since p is odd and $s'(z) \neq 0$ for $z \neq 0$, this segment is the half-line $-1 < w < \infty$. So $z=1$ is the only singularity of $s(z)$ for $|z| \leq 1$. By the Schwarz reflection principle we can continue $s(z)$ across the circle $|z|=1$. The reflection of S across the line $\Re(w) = -1$ shows that $s(z)$ maps $|z| > 1$ on a region S^* consisting of $p-1$ full planes and a half-plane $\Re(w) < -1$ all tied at $w = -2$, the image of $z = \infty$. Thus $s(z)$ is regular in the entire complex plane with the exception of the point $z=1$, and maps the plane on a region consisting of $2p-1$ sheets. Therefore $s(z)$ is a rational function of degree $2p-1$, and since $s(z)$ takes every value $2p-1$ times, $z=1$ is a pole of order $2p-1$ and

$$(5.21) \quad s(z) = \frac{a_p z^p + a_{p+1} z^{p+1} + \cdots + a_{2p-1} z^{2p-1}}{(1-z)^{2p-1}}.$$

On the other hand, consideration of S^* shows that $s_1(z) = -2 - s(1/z)$ also maps $|z| < 1$ onto S and takes $-1 < z < 1$ onto the half-line $-1 < w < \infty$. Hence $s(z) = s_1(z)$ or $s(z) + s(1/z) = -2$. Using this with (5.21) gives

$$(5.22) \quad s(z) = \frac{-2 \sum_{n=p}^{2p-1} (-1)^n C_{2p-1, n} z^n}{(1-z)^{2p-1}}.$$

But except for a magnification the image of E for $s(z)$ is the same as that given by (5.19). Therefore for p odd

$$(5.23) \quad f_M(z) = p \int_0^z \frac{t^p}{(1-t)^{2p}} dt = \frac{(-1)^p \sum_{n=p}^{2p-1} (-1)^n C_{2p-1, n} z^n}{C_{2p-1, p} (1-z)^{2p-1}}.$$

A similar argument can be given to show that (5.23) also holds for p even.

The form of the function (5.18) can also be obtained directly from the mapping properties. For suppose $s(z)$ maps E onto a region S consisting of $p-1$ full planes and a plane with a radial slit along the real axis. The z -plane is rotated so that $s(1) = \infty$. The function is to have a q th order root at $z=0$ ($q \geq 1$) and a $(p-q)$ th order root at $z = -\rho$, $0 < \rho < 1$. Finally we may require that $s^{(q)}(0) > 0$, which because of the symmetry of S implies that the interval $0 \leq z < 1$ goes into $0 \leq w < \infty$. Reflection across the radial slit gives S^* just a duplication of S , and $s(z) = s(1/z)$. Just as before $s(z)$ is a rational function of degree $2p$, with a pole of order $2p$ at $z=1$ and a root of order $p-q$ at $-1/\rho$. So

$$(5.24) \quad s(z) = c \frac{z^q(1 + z/\rho)^{p-q}(1 + \rho z)^{p-q}}{(1 - z)^{2p}} u(z),$$

where $u(z)$ is a polynomial of degree not greater than q . Using $s(z) = s(1/z)$ with (5.24) yields $u(z) = 1$ and $s(z)$ is identical with (5.18) when $c = 1$. A similar argument can be given to obtain (5.16).

THEOREM 3. Let $F(z) \in S(p)$ of the form (5.10) have $p - q$ roots $\beta_1, \beta_2, \dots, \beta_{p-q}$ such that $0 < |\beta_j| \leq \rho < 1$ for $j = 1, 2, \dots, p - q$. Then for $\rho \leq r \leq 1$

$$(5.25) \quad \begin{aligned} |F(re^{i\theta})| &\geq \frac{r^q}{(1 + r)^{2p}} \prod_{j=1}^{p-q} (1 - r|\beta_j|) \left(\frac{r}{|\beta_j|} - 1 \right) \\ &\geq \frac{r^q(1 - \rho r)^{p-q}(r/\rho - 1)^{p-q}}{(1 + r)^{2p}}. \end{aligned}$$

The bounds are sharp, the first equality occurring for the function (5.16) and the second for the function (5.18) when $z = -r$.

Proof. We first note that for $|\beta_j| \leq \rho \leq r = |z| \leq 1$, $|(1 - z/\beta_j)(1 - \bar{\beta}_j z)| = |(\beta_j - z)(\bar{\beta}_j^{-1} - \bar{z})\bar{\beta}_j/\beta_j| \geq (r - |\beta_j|)(|\beta_j|^{-1} - r)$, with equality if and only if $\arg z = \arg \beta_j$. Using Lemma 1 and Lemma 2 in an obvious fashion the theorem follows at once.

COROLLARY. If $F(z) = z^p + \dots \in S(p)$, then

$$(5.26) \quad |F(re^{i\theta})| \geq \frac{r^p}{(1 + r)^{2p}}, \quad 0 \leq r \leq 1,$$

and this inequality is sharp.

This can also be obtained directly from the theorem for univalent functions.

THEOREM 4. Let $f(z) = z + \dots \in C(p)$ have critical points $\beta_1, \beta_2, \dots, \beta_{p-1} \neq 0$ in E . Let r_1 be the least positive root of

$$(5.27) \quad 0 = 1 - r \left[\frac{2p}{1 + r} + \sum_{j=1}^{p-1} \frac{1}{|\beta_j| - r} + \frac{|\beta_j|}{1 - r|\beta_j|} \right] = J_1(r).$$

Then $f(rz)/r \in C(1)$ for every r , $0 \leq r \leq r_1$. The function

$$(5.28) \quad f_M(z) = \int_0^z \frac{1}{(1 + t)^{2p}} \prod_{j=1}^{p-1} \left(1 - \frac{t}{|\beta_j|} \right) (1 - t|\beta_j|) dt$$

shows that the upper bound r_1 cannot be increased.

Proof. Let $\{f_m(z)\}$ be a sequence of functions of the form (5.9) with $q = 1$ which converges to $f(z)$. For each $f_m(z)$

$$(5.29) \quad 1 + \frac{zf_m''(z)}{f_m'(z)} = 1 + \sum_{i=1}^m \gamma_i \frac{zz_i}{1-zz_i} - \sum_{i=1}^{p-1} \frac{z}{\beta_i - z} + \frac{z\bar{\beta}_i}{1-z\bar{\beta}_i}.$$

For $|z| = r < 1$, $\Re(zz_i/(1-zz_i)) \geq -r/(1+r)$. For $|z| \leq r < \min \{|\beta_1|, |\beta_2|, \dots, |\beta_{p-1}|\}$,

$$\begin{aligned} \left| \Re \sum_{i=1}^{p-1} \frac{z}{\beta_i - z} + \frac{z\bar{\beta}_i}{1-z\bar{\beta}_i} \right| &\leq \left| \sum_{i=1}^{p-1} \frac{z}{\beta_i - z} + \frac{z\bar{\beta}_i}{1-z\bar{\beta}_i} \right| \\ &\leq \sum_{i=1}^{p-1} \frac{r}{|\beta_i| - r} + \frac{r|\beta_i|}{1-r|\beta_i|}. \end{aligned}$$

Therefore

$$(5.30) \quad 1 + \Re \left(\frac{zf_m''(z)}{f_m'(z)} \right) \geq J_1(r) \geq 0$$

for $0 \leq |z| \leq r_1$.

THEOREM 5. Let $F(z) = z + \dots \in S(p)$ have roots $\beta_1, \beta_2, \dots, \beta_{p-1} \neq 0$ in E . Let r_1 be defined as in Theorem 4. Then $F(rz)/r \in S(1)$ for every r , $0 \leq r \leq r_1$.

Proof. Let $f(z)$ be defined by (5.5). Then for each sequence $\{f_m(z)\}$ converging to $f(z)$, the sequence $\{F_m(z) = zf_m'(z)\}$ converges to $F(z)$. Finally

$$\Re \left(z \frac{F_m'(z)}{F_m(z)} \right) = 1 + \Re \left(z \frac{f_m''(z)}{f_m'(z)} \right) \geq J_1(r) \geq 0$$

for $0 \leq |z| \leq r_1$.

Theorems 4 and 5 are special cases of Theorems 6 and 7 respectively. The proofs are similar and so are omitted.

THEOREM 6. Let $f(z) = z^q + \dots \in C(p)$ have critical points $\beta_1, \beta_2, \dots, \beta_{p-q} \neq 0$ in E . Let r_q be the least positive root of

$$(5.31) \quad 0 = q - r \left[\frac{2p}{1+r} + \sum_{i=1}^{p-q} \frac{1}{|\beta_i| - r} + \frac{|\beta_i|}{1-r|\beta_i|} \right] = J_q(r).$$

Then $f(rz)/r^q \in C(q)$ for every r in the interval $0 \leq r \leq r_q$. The function (5.13) shows that the upper bound r_q cannot be increased.

THEOREM 7. Let $F(z) = z^q + \dots \in S(p)$ have roots $\beta_1, \beta_2, \dots, \beta_{p-q} \neq 0$ in E . Let r_q be defined as in Theorem 6. Then $F(rz)/r^q \in S(q)$ for every r , $0 \leq r \leq r_q$. The function (5.16) shows that the upper bound r_q cannot be increased.

Since $J_q(r)$ is a decreasing function of $|\beta_j|$ for $r < |\beta_j| < 1$, we obtain bounds for r_q by solving the cubic

$$(5.32) \quad 0 = q - r \left[\frac{2p}{1+r} + (p-q) \left(\frac{1}{y-r} + \frac{y}{1-ry} \right) \right]$$

for its least positive root. This gives

$$(5.33) \quad \rho(y) = \frac{p(1+y)^2 - 2qy - (1+y)(p^2(1+y)^2 - 4pqy)^{1/2}}{2qy},$$

so that if $0 < m \leq |\beta_j| \leq M < 1$ for $j = 1, 2, \dots, p-q$, then

$$(5.34) \quad \frac{qm}{p(1+m)^2} \leq \rho(m) \leq r_q \leq \rho(M).$$

6. The coefficient problem. It has recently⁽²⁰⁾ been conjectured that if

$$(6.1) \quad F(z) = \sum_{n=1}^{\infty} b_n z^n$$

is p -valent in E , then for $n = p+1, p+2, \dots$

$$(6.2) \quad |b_n| \leq \sum_{k=1}^p \frac{2k(n+p)!}{(n^2 - k^2)(p+k)!(p-k)!(n-p-1)!} |b_k|.$$

For $p=2, n=3$, this gives the conjecture that

$$(6.3) \quad |b_3| \leq 5|b_1| + 4|b_2|.$$

THEOREM 8. *Let $F(z) \in S(2)$ have the form (6.1), and let all the coefficients b_n be real. Then (6.3) is valid and this inequality is sharp for every pair $|b_1|, |b_2|$, not both zero.*

Proof. We may assume $|b_1| \neq 0$, for if $b_1 = 0$ (6.3) is a special case of Theorem 2 with $p=q=2$. By Lemma 1 and Lemma 2, we have a sequence of functions $F_m(z)$ of the form

$$(6.4) \quad \begin{aligned} F_m(z) &= c_1 z \left(1 - \frac{z}{\beta} \right) (1 - \bar{\beta}z) \prod_{j=1}^m (1 - z_j z)^{-\gamma_j} \\ &= c_1 z \left(1 - \frac{z}{\beta} \right) (1 - \bar{\beta}z) \sum_{n=0}^{\infty} A_n z^n = \sum_{n=1}^{\infty} c_n z^n, \end{aligned}$$

where

$$(6.5) \quad \sum_{j=1}^m \gamma_j = 4, \quad 0 < \gamma_j, \quad z_j = \cos \theta_j + i \sin \theta_j.$$

The sequence $F_m(z)$ converges to $F(z)$ and $c_n \rightarrow b_n$ as $m \rightarrow \infty$. Since $b_1 \neq 0$,

⁽²⁰⁾ On some determinants related to p -valent functions, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 175-192.

$\beta \neq 0$, and since all coefficients are real, the single critical point must lie on the real axis, that is $\beta = \bar{\beta}$. From (6.4)

$$\begin{aligned} A_0 &= 1, \\ (6.6) \quad A_1 &= \sum_{j=1}^m \gamma_j z_j, \\ A_2 &= \frac{1}{2} \sum_{j=1}^m \gamma_j z_j^2 + \frac{1}{2} \left(\sum_{j=1}^m \gamma_j z_j \right)^2, \end{aligned}$$

and

$$\begin{aligned} (6.7) \quad c_2 &= A_1 c_1 - c_1(\beta + \beta^{-1}), \\ c_3 &= A_2 c_1 - A_1 c_1(\beta + \beta^{-1}) + c_1, \end{aligned}$$

from which

$$(6.8) \quad c_3 = c_1(1 + A_2 - A_1^2) + A_1 c_2.$$

It is clear from the conditions (6.5) that $|A_1| \leq 4$. We need only prove that $|1 + A_2 - A_1^2| \leq 5$. Since all the coefficients are real, the image of E under $f(z)$ will be symmetric about the real axis. In selecting a polygon for approximation, we may require that this polygon is also symmetric about the real axis. Then in the sums (6.6), the z_j occur in conjugate pairs and each element of a conjugate pair is multiplied by the same γ_j . So

$$\begin{aligned} (6.9) \quad A_2 - A_1^2 &= \frac{1}{2} \sum_{j=1}^m \gamma_j z_j^2 - \frac{1}{2} \left(\sum_{j=1}^m \gamma_j z_j \right)^2 \\ &= \frac{1}{2} \sum_{j=1}^m \gamma_j (2 \cos^2 \theta_j - 1) - \frac{1}{2} \left(\sum_{j=1}^m \gamma_j \cos \theta_j \right)^2 \\ &= -2 + \sum_{j=1}^m \gamma_j \cos^2 \theta_j - \frac{1}{2} \left(\sum_{j=1}^m \gamma_j \cos \theta_j \right)^2. \end{aligned}$$

By Cauchy's inequality,

$$\sum_{j=1}^m \gamma_j \cos^2 \theta_j - \frac{1}{4} \left(\sum_{j=1}^m \gamma_j \cos \theta_j \right)^2 \geq 0,$$

so, using (6.5), we have $-6 \leq A_2 - A_1^2 \leq 2$ or

$$(6.10) \quad -5 \leq 1 + A_2 - A_1^2 \leq 3.$$

We shall see when we have proved Theorem 12 that (6.3) is sharp for every pair $|b_1|$, $|b_2|$, and that the inequalities of Theorems 9, 10, and 11 are also sharp in the same sense.

Using Lemma 1 and Theorem 8 we have immediately the following theorem.

THEOREM 9. Let $f(z) \in C(2)$ have the form

$$(6.11) \quad f(z) = \sum_{n=1}^{\infty} a_n z^n$$

and let all coefficients be real. Then

$$(6.12) \quad |a_3| \leq \frac{5}{3} |a_1| + \frac{8}{3} |a_2|,$$

and this inequality is sharp for every pair $|a_1|, |a_2|$ not both zero.

Notice that the conjecture (6.2) for p -valent functions suggests the conjecture that

$$(6.13) \quad |a_n| \leq \sum_{k=1}^p \frac{2k^2(n+p)!}{n(n^2-k^2)(p+k)!(p-k)!(n-p-1)!} |a_k|$$

for functions of class $C(p)$ of the form (6.11). This of course gives (6.12) for $p=2$ and $n=3$. It also seems reasonable to conjecture that the bounds obtained in Theorems 2, 3, and 7 are valid for all functions regular and p -valent in E .

The same methods yield the following extensions of Theorems 8 and 9.

THEOREM 10. Let $F(z) \in S(p)$ be of the form (6.1) with $b_1=b_2=\dots=b_{p-2}=0$, and let all coefficients be real. Then

$$(6.14) \quad |b_{p+1}| \leq (2p+1)(p-1) |b_{p-1}| + 2p |b_p|,$$

and this inequality is sharp for every pair $|b_{p-1}|, |b_p|$ not both zero.

THEOREM 11. Let $f(z) \in C(p)$ be of the form (6.11) with $a_1=a_2=\dots=a_{p-2}=0$, and let all the coefficients be real. Then

$$(6.15) \quad |a_{p+1}| \leq \frac{(2p+1)(p-1)^2}{p+1} |a_{p-1}| + \frac{2p^2}{p+1} |a_p|,$$

and this inequality is sharp for every pair $|a_{p-1}|, |a_p|$ not both zero.

The inequalities (6.14) and (6.15) are special cases of the conjectures (6.2) and (6.13) respectively.

It has been shown⁽²⁰⁾ that for every set $|b_1|, |b_2|, \dots, |b_p|$ not all zero, there is a p -valent function with b_n satisfying (6.2) with the equality sign. All of these p -valent functions are of the type described in the following theorem and so are of class $S(p)$. From these we can obtain functions of

class $C(p)$ for which the equality sign holds in (6.13) for every set $|a_1|, |a_2|, \dots, |a_p|$ not all zero.

THEOREM 12. *Let*

$$(6.16) \quad f(z) = P(u) = \sum_{k=1}^p (-1)^k a_k u^k, \quad u = \frac{z}{(1-z)^2},$$

where $a_k \geq 0$, $k=1, 2, \dots, p$, and for at least one k , $a_k > 0$. Then $f(z)$ is an element of class $S(p)$ and, moreover, maps E onto a region R consisting of $p-1$ full planes and a plane with a single radial slit.

Proof. It will be simpler and completely equivalent to consider $g(z) = f(-z)$. Then

$$(6.17) \quad g(z) = a_1 v + a_2 v^2 + \dots + a_p v^p, \quad v = \frac{z}{(1+z)^2},$$

and for $g(z)$

$$(6.18) \quad \begin{aligned} H(r, \theta) &= \Re \left(z \frac{g'(z)}{g(z)} \right) = \Re \left(\frac{a_1 + 2a_2 v + \dots + p a_p v^{p-1}}{a_1 + a_2 v + \dots + a_p v^{p-1}} \frac{z}{v} \frac{dv}{dz} \right) \\ &= \Re \left(Q(v) \frac{z}{v} \frac{dv}{dz} \right). \end{aligned}$$

Now if $z = e^{i\theta}$, then $v \geq 1/4$, $Q(v) > 0$, and

$$(6.19) \quad T(z) = \frac{z}{v} \frac{dv}{dz} = \frac{1-z}{1+z} = i \frac{\cos \theta - 1}{\sin \theta}.$$

Therefore $G(1, \theta) = \Re(Q(v)T(e^{i\theta})) = 0$, and the boundary of R consists of radial slits. Since $g'(z)$ has only a single simple root on $|z| = 1$, there is only a single slit.

The function $h(z) = zg'(z)/g(z)$ is regular in some ring domain $\rho < |z| < 1$. To determine $\Re(h(z))$ for $|z| < 1$, we examine more closely $h(z)$ for $|z| = 1$.

A simple computation shows that

$$(6.20) \quad Q'(v) = \frac{\sum_{\mu \geq p-1}^p a_\mu a_\nu (\mu - \nu) 2v^{\mu+\nu-3}}{(a_1 + a_2 v + \dots + a_p v^{p-1})^2}.$$

If only one coefficient a_k is different from zero, then $Q(v)$ is a positive constant. Otherwise $Q'(v) > 0$ for $v \geq 1/4$.

Let $z = e^{i\theta}$ and let θ vary from 0 to π . Then $T(z)$ runs along the imaginary axis from 0 to $-i\infty$, $v(e^{i\theta})$ runs along the real axis from $1/4$ to $+\infty$, and $Q(v)$ is either a positive constant or a monotonically increasing positive func-

tion. Thus $h(z)$ maps this arc of the unit circle in a one-to-one manner on the negative imaginary axis, a counterclockwise direction for z , corresponding to a downward direction for $h(z)$. As θ runs from π to 2π , $T(z)$ runs along the imaginary axis from $+\infty$ to $1/4$, and $Q(v)$ is either a positive constant or a monotonically decreasing positive function. Thus $h(z)$ maps this arc in a one-to-one manner on the positive imaginary axis, with the same correspondence of directions as before. Since a regular function is region-preserving there exists a ρ such that for $\rho < |z| < 1$

$$H(r, \theta) = \Re(h(z)) > 0,$$

and hence $g(z) \in S(p)$.

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